The Quadric Spreads Connected with the Configuration $\Gamma^{n+2}_{n+4,n}$, and a Special Case in the Pascal Hexagram.

By W. B. Carver.

In a paper on the Cayley-Veronese Configurations * the author called attention to six conics connected with the configuration $\Gamma^4_{6,2}$. This configuration contains six $\Gamma^3_{5,2}$'s (a $\Gamma^3_{5,2}$ is the well-known Desargues configuration), and with each $\Gamma^3_{5,2}$ is connected a conic ϕ whose polar system sends each point of the configuration into its corresponding line. These six conics lie in a pencil, *i. e.*, they pass through four points. This theorem was proven in the previous paper by a synthetic method which was not capable of immediate extension to the n-dimensional case. The first object of the present paper will be to give an analytic proof for the plane configuration, and then, by a simple extension, to obtain similar theorems for the n-dimensional configuration $\Gamma^{n+2}_{n+4,n}$. In the second part of this paper certain of these pencils of conics connected with the Pascal hexagram will be considered.

I. The Quadric Spreads of the $\Gamma^{n+2}_{n+4,n}$.

1. *The Plane Configuration* $\Gamma^4_{6,2}$. We may take as the equations of any six points in S_4

$$\xi_i = 0 \qquad (i = 1, 2, \ldots, 6)$$

with the relation

$$\sum_i \xi_i = 0$$

Any three of these points, as $\xi_1 = 0$, $\xi_2 = 0$, and $\xi_3 = 0$, determine a plane, 123; and there are twenty such planes. When we take a plane section of this figure

* *Transactions of the American Mathematical Society*, Vol. VI, pp. 534–545 (October, 1905). The notation used in this present paper was defined in the earlier paper.

we obtain the plane configuration $\Gamma^4_{6,2}$, and the plane ijk of the 4-dimensional figure gives the point ijk* of the configuration $\Gamma^4_{6,2}$. Let the equations of the cutting plane be

$$\begin{cases} \sum_i \alpha_i x_i = 0 \\ \sum_i \beta_i x_i = 0 \end{cases} \qquad (i = 1, 2, \ldots, 5)$$

Then the equations of the twenty points of the $\Gamma^4_{6,2}$ will be

$$\begin{vmatrix} \xi_i & \xi_j & \xi_k \\ \alpha_i & \alpha_j & \alpha_k \\ \beta_i & \beta_j & \beta_k \end{vmatrix} = 0 \qquad (i, j, k = 1, 2, \ldots, 6)$$

with the relations

$$\sum_i \xi_i = 0$$

$$\sum_i \alpha_i = 0$$

$$\sum_i \beta_i = 0$$

(The last two relations may be regarded as definitions of α_6 and β_6.)

If we take the coefficients of the ξ's in these equations as co-ordinates of the points we have three superfluous co-ordinates. These superfluous co-ordinates are conducive to symmetry, but they are very inconvenient in determining the equations of the conics ϕ. Hence, sacrificing symmetry to some extent, we eliminate ξ_6 by using the relation

$$\sum_i \xi_i = 0$$

and then simply drop the 4th and 5th co-ordinates, leaving three ordinary homogeneous co-ordinates for each point.† These, together with the co-ordinates dropped, are shown in the table which follows.

* *Loc. cit.*, p. 535.

† Discarding the 4th and 5th co-ordinates of these points amounts to projecting them from the line 45 upon the plane 123 of the reference figure in S_4.

$$\pi_{ij} \equiv \begin{vmatrix} \alpha_i & \alpha_j \\ \beta_i & \beta_j \end{vmatrix}$$

Points.	Co-ordinates.				
123	π_{23}	π_{31}	π_{12}	0	0
456	π_{45}	π_{45}	π_{45}	$\pi_{45}+\pi_{65}$	$\pi_{45}+\pi_{46}$
234	0	π_{34}	π_{42}	π_{23}	0
314	π_{43}	0	π_{14}	π_{31}	0
124	π_{24}	π_{41}	0	π_{12}	0
235	0	π_{35}	π_{52}	0	π_{23}
315	π_{53}	0	π_{15}	0	π_{31}
125	π_{25}	π_{51}	0	0	π_{12}
145	π_{45}	0	0	π_{51}	π_{14}
245	0	π_{45}	0	π_{52}	π_{24}
345	0	0	π_{45}	π_{53}	π_{34}
146	$\pi_{14}+\pi_{64}$	π_{14}	π_{14}	$\pi_{14}+\pi_{16}$	π_{14}
246	π_{24}	$\pi_{24}+\pi_{64}$	π_{24}	$\pi_{24}+\pi_{26}$	π_{24}
346	π_{34}	π_{34}	$\pi_{34}+\pi_{64}$	$\pi_{34}+\pi_{36}$	π_{34}
156	$\pi_{15}+\pi_{65}$	π_{15}	π_{15}	π_{15}	$\pi_{15}+\pi_{16}$
256	π_{25}	$\pi_{25}+\pi_{65}$	π_{25}	π_{25}	$\pi_{25}+\pi_{26}$
356	π_{35}	π_{35}	$\pi_{35}+\pi_{65}$	π_{35}	$\pi_{35}+\pi_{36}$
236	π_{23}	$\pi_{23}+\pi_{63}$	$\pi_{23}+\pi_{26}$	π_{23}	π_{23}
316	$\pi_{31}+\pi_{36}$	π_{31}	$\pi_{31}+\pi_{61}$	π_{31}	π_{31}
126	$\pi_{12}+\pi_{62}$	$\pi_{12}+\pi_{16}$	π_{12}	π_{12}	π_{12}

Now the conic ϕ_i is such that its polar system sends any point of the configuration whose symbol contains the digit i into the line whose symbol contains the digit i together with the three digits not contained in the symbol of the point.* Thus ϕ_1 sends the point 123 into the line 1456, ϕ_6 sends this line 1456 into the point 236, etc. Since the line 1456 contains the points 456, 156, 146, and 145, it is evident that any one of these four points together with the point 123 forms a conjugate pair with respect to ϕ_1. Five independent pairs of conjugate points determine the conic, and its equation may be written at once in terms of the co-ordinates of these points. Having treated 4 and 5 in a special manner, we do not expect the equations of ϕ_4 and ϕ_5 to be symmetrical with

* *Loc. cit.*, p. 543.

those of ϕ_1, ϕ_2, ϕ_3, and ϕ_6. We obtain the following sufficiently simple results:

$$\phi_4 \equiv \sum_i \pi_{i4}\,(\pi_{i4}+\pi_{54})\,x_i^2 + 2\sum_{ij}\pi_{i4}\,\pi_{j4}\,x_i\,x_j = 0$$

$$\phi_5 \equiv \sum_i \pi_{i5}\,(\pi_{i5}+\pi_{45})\,x_i^2 + 2\sum_{ij}\pi_{i5}\,\pi_{j5}\,x_i\,x_j = 0 \qquad (i, j = 1, 2, 3)$$

$$\phi_k \equiv \sum_i \left[\pi_{k4}\,\pi_{i5}\,(\pi_{i5}+\pi_{45}) + \pi_{k5}\,\pi_{i4}\,(\pi_{i4}+\pi_{54})\right] x_i^2$$

$$+\, 2\sum_{ij}(\pi_{k4}\,\pi_{i5}\,\pi_{j5} + \pi_{k5}\,\pi_{i4}\,\pi_{j4})\,x_i\,x_j \equiv \pi_{k4}\,\phi_5 + \pi_{k5}\,\phi_4 = 0 \qquad (k = 1, 2, 3, 6)$$

The fact that the six conics lie in a pencil is evident at once from these equations.

[The π's may be regarded as co-ordinates in S_4 of the cutting plane, an extension of Plücker's line co-ordinates in ordinary space. There are altogether $\binom{6}{2}$ or 15 of them. Between these 15 π's, however, there are six independent* quadratic relations of the type

$$\pi_{12}\,\pi_{34} + \pi_{13}\,\pi_{42} + \pi_{14}\,\pi_{23} = 0$$

In finding the equations of the ϕ's, these six relations have been used to eliminate six of the π's. Hence these equations are expressed in terms of nine π's, between which there are only the two linear relations

$$\Sigma\,\pi_{i4} = 0$$

and

$$\Sigma\,\pi_{i5} = 0 \qquad (i = 1, 2, \ldots, 6)$$

If we use these two relations to eliminate π_{64} and π_{65}, the equations of the ϕ's will contain only seven homogeneous, or six non-homogeneous, independent constants—the proper number to fix a plane in S_4.]

2. *The n-Dimensional Case.* The $\binom{n+4}{3}$ points of the configuration $\Gamma^{n+2}_{n+4,\,n}$ may be given by the equations

$$\begin{vmatrix} \xi_i & \xi_j & \xi_k \\ \alpha_i & \alpha_j & \alpha_k \\ \beta_i & \beta_j & \beta_k \end{vmatrix} = 0 \qquad (i, j, k = 1, 2, \ldots, n+4)$$

with the relations

$$\sum_i \xi_i = 0$$

$$\sum_i \alpha_i = 0$$

$$\sum_i \beta_i = 0$$

* There are altogether $\binom{6}{4}$ or 15 such relations. We obtain a set of six which are independent by selecting the six which contain a particular π, as π_{12}.

With this configuration are connected $n+4$ quadric spreads. The spread ϕ_i is defined by the fact that its polar system sends any point whose symbol contains the digit i into the co-point (or S_{n-1}) whose symbol contains the digit i and the $n+1$ digits not contained in the symbol of the point.* The equations of these quadrics are

$$\phi_{n+2} \equiv \sum_i \pi_{i(n+2)} \left(\pi_{i(n+2)} + \pi_{(n+3)\,(n+2)}\right) x_i^2 + 2 \sum_{ij} \pi_{i(n+2)}\, \pi_{j(n+2)}\, x_i\, x_j = 0$$

$$\phi_{n+3} \equiv \sum_i \pi_{i(n+3)} \left(\pi_{i(n+3)} + \pi_{(n+2)\,(n+3)}\right) x_i^2 + 2 \sum_{ij} \pi_{i(n+3)}\, \pi_{j(n+3)}\, x_i\, x_j = 0$$

$$\phi_k \quad \equiv \pi_{k(n+2)}\, \phi_{n+3} + \pi_{k(n+3)}\, \phi_{n+2} = 0$$

$$(i, j = 1, 2, \ldots, n+1 \quad \text{and} \quad k = 1, 2, \ldots, n+1, n+4)$$

It is evident that these $n+4$ quadrics lie in a pencil, *i. e.*, that they all pass through an $(n-2)$-way spread of the 4th degree.† The equations contain $2n+5$ π's which are connected only by the two linear relations

$$\sum_i \pi_{i(n+2)} = 0$$

$$\sum_i \pi_{i(n+3)} = 0$$

We have then $2n+3$ homogeneous, or $2n+2$ non-homogeneous, independent constants.

3. *The* $\Gamma^{n+2}_{n+4,\,n}$ *Determined by the Quadrics.* If to the $2n+2$ constants in our equations of the quadrics we add the n^2+2n constants of a collineation in S_n, we have n^2+4n+2, the proper number of constants for the *general* pencil of $n+4$ quadrics in S_n. The configuration $\Gamma^{n+2}_{n+4,\,n}$ is also determined by n^2+4n+2 arbitrary constants.‡ This suggests that the pencil of $n+4$ quadrics may be taken arbitrarily, and that the configuration with which they are connected will then be determined. This may be verified analytically.

Consider, for simplicity, the plane case. The polar system of the conic ϕ_1 sends the point 123 into the line 1456, etc. Hence, using the ϕ's as operators, and operating on the point 123 successively with ϕ_1, ϕ_4, ϕ_2, ϕ_5, and ϕ_3, it is sent

---

* Cf. Veronese, Behandlung der projectivischen Verhältnisse der Räume von verschiedenen Dimensionen, etc., *Mathematische Annalen*, Vol. XIX (1882), p. 194.

† The simple one-dimensional case of this theorem is interesting, and is, to the best of the writer's knowledge, new. The $\Gamma^3_{5,1}$ is the configuration of ten points on a straight line obtained by making an arbitrary line-section of the Desargues figure. It is well known that we can pick out of these ten points (in five different ways) a set of six which form three pairs of a quadratic involution. Our theorem gives the additional fact that *the five pairs of double points of these involutions are pairs of another involution.*

‡ Author's paper, *loc. cit.*, Theorem IX.

into the line 1236, and this line passes through the point 123. The equations of the ϕ's being known, this gives a quadratic relation which the co-ordinates of the point 123 must satisfy. If now we replace ϕ_4 and ϕ_5 by ϕ_5 and ϕ_6 respectively, the point 123 will be sent into the line 1234; and this gives a second similar relation. These two equations are sufficient to determine the point 123; but since they are quadratic equations, there will be four solutions. Having thus determined the point 123 (or any other point of the configuration) the remainder of the configuration is *uniquely* determined, being readily built up by the *linear* process of taking poles and polars with respect to the ϕ's. A given pencil of six conics may then be connected with any one of *four* $\Gamma_{6,2}^{4}$ configurations which are determined by the conics.

A similar procedure serves for any $\Gamma_{n+4,n}^{n+2}$. The quadrics $\phi_1, \phi_4, \phi_2, \phi_5$, and ϕ_3 will send the point 123 into the co-point (or S_{n-1}) $12367 \ldots (n+4)$, giving a quadratic relation which must be satisfied by the co-ordinates of the point 123. Replacing ϕ_4 and ϕ_5 by ϕ_5 and ϕ_6 respectively, then ϕ_5 and ϕ_6 by ϕ_6 and $\phi_7, \ldots$ and finally ϕ_{n+2} and ϕ_{n+3} by ϕ_{n+3} and ϕ_{n+4}, we obtain, in all, n equations,* which are sufficient to determine the point 123. Since they are quadratic relations, there will be 2^n solutions. Hence, a given pencil of $n+4$ quadric spreads in S_n may be connected with any one of 2^n $\Gamma_{n+4,n}^{n+2}$ configurations which are determined by the quadrics.

4. *Quadric Spreads Connected with the General Configuration* $\Gamma_{n,r}^{\nu}$. The theorems in the author's previous paper† concerning the conics connected with the general plane configuration $\Gamma_{n,2}^{\nu}$ may now be readily extended. We have first the following:

Connected with every $\Gamma_{n,r}^{\nu}$ (*where* $\nu \geqq r+2$ *and* $n \geqq \nu+2$) *are* $\binom{n}{n-r-3}\binom{n-r-3}{\nu-r-1}$ *quadrics lying by* $(r+4)$*'s in* $\binom{n}{n-r-4}\binom{n-r-4}{\nu-r-2}$ *pencils, each quadric in* $\nu-r-1$ *pencils.*

And combining this theorem with its dual, we have:

Connected with every $\Gamma_{n,r}^{\nu}$ (*where* $\nu \geqq r+2$ *and* $n \geqq \nu+3$) *are* $\binom{n}{n-r-3}\binom{n-r-3}{\nu-r-1}$ *quadrics which lie by* $(r+4)$*'s in* $\binom{n}{n-r-4}\binom{n-r-4}{\nu-r-2}$ *pencils, each quadric in* $\nu-r-1$ *pencils; and which also lie by* $(r+4)$*'s in* $\binom{n}{n-r-4}\binom{n-r-4}{\nu-r-1}$ *ranges, each quadric in* $n-\nu-2$ *ranges.*

* There are, in fact, $\binom{n+1}{2}$ such conditions on the point 123; but n of them, *which are independent*, may be chosen in the manner indicated.

† *Loc. cit.*, pp. 544, 545.

II. Special $\Gamma^4_{6,2}$'s in the Pascal Hexagram.

1. *Cayley's Special* $\Gamma^4_{6,2}$. There are eleven examples of the configuration $\Gamma^4_{6,2}$ in the Pascal hexagram, of which the best known and least interesting is made up of the twenty Steiner points and fifteen Plücker lines.* The remaining ten examples were treated by Cayley.† Let a, b, c, d, e and f be six points on a conic C. Consider the six hexagons obtained by taking a, b and c as alternate vertices and permuting d, e and f in the six possible ways for the other vertices. The six corresponding Pascal lines meet by threes in two Steiner points. These six Pascal lines and the nine lines which make up the sides of the six hexagons are the fifteen lines of a $\Gamma^4_{6,2}$. (See Fig. 1. The Pascal lines are dotted, and the other nine lines are solid.) But these lines meet by threes, not only in the twenty points of the $\Gamma^4_{6,2}$, but also in six extra points—the points on the conic C. The lines and points of the $\Gamma^4_{6,2}$ are designated by the usual combinations of the digits 1, 2, ..., 6. The two Steiner points, being opposite points of the configuration, have the symbols 123 and 456. The three lines 2345, 1356 and 1246, which in the general $\Gamma^4_{6,2}$ would not be collinear, meet in the point a. The two digits not contained in the symbol of each of these three lines are respectively 16, 24 and 35; each pair being composed of one of the digits 1, 2, 3, and one of the digits 4, 5, 6. Paring them off differently, as 15, 26 and 34, we are led to the three lines 2346, 1345 and 1256 which meet in the point b. We may similarly find each of the six sets of three lines which meet at each of the points a, b, ..., f. They are shown in the following table:

	a	b	c	d	e	f
1	6: 2345	5: 2346	4: 2356	6: 2345	4: 2356	5: 2346
2	4: 1356	6: 1345	5: 1346	5: 1346	6: 1345	4: 1356
3	5: 1246	4: 1256	6: 1245	4: 1256	5: 1246	6: 1245

* For a bibliography of the Pascal hexagram and a résumé of the known theorems, see Salmon, *Conic Sections*, p. 379; Richmond, On the Figure of Six Points in Space of Four Dimensions, *Quarterly Journal*, Vol. XXXI (1899), p. 125; Klug, Die Configuration des Pascal'schen Sechseckes (1898).

† Sur Quelques Théorèmes de la Géométrie de Position, *Crelle's Journal*, Vol. XXXI (1846); or *Collected Papers*, Vol. I, p. 317. Also, On Pascal's Theorem, *Quarterly Journal*, Vol. IX (1868); or *Collected Papers*, Vol. VI, p. 129.

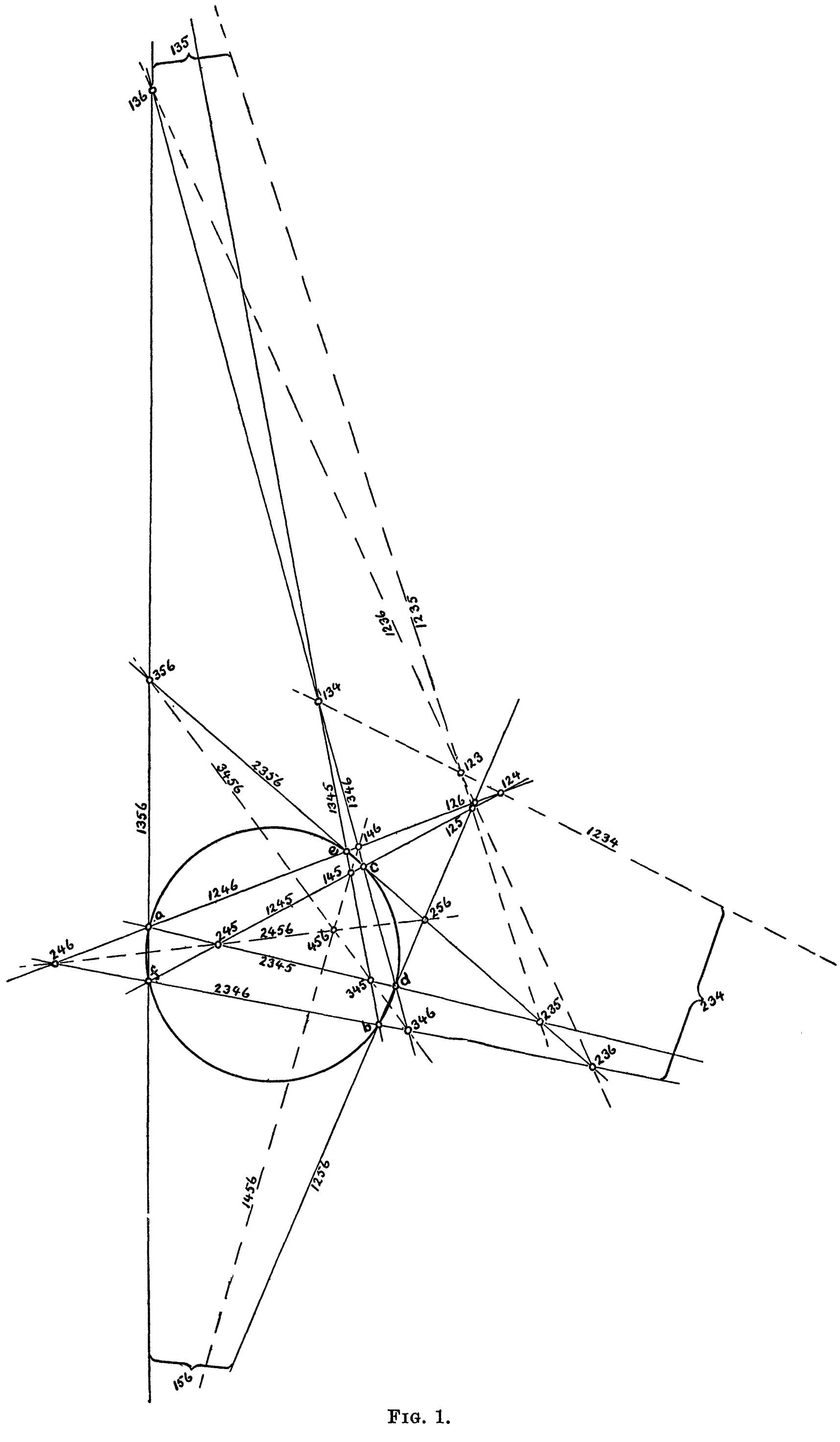
135
136
1356
1236
1235
356
134
123
124
126
125
1234
3456
2356
1345
1346
146
e
c
145
1246
1245
a
256
245
2456
456
246
f
2345
345
d
2346
235
b
346
236
234
1456
1256
156

FIG. 1.

2. *Analytic Methods.* Since we have the co-ordinates of the points of the general $\Gamma^4_{6,2}$, the equations of its lines may be written, and we may derive at once the conditions that the required sets of lines may be collinear. Thus the lines

$$\begin{array}{ll} 2345 & x_1 = 0 \\ 1356 & -\pi_{15}x_1 + (\pi_{15} + \pi_{35} + \pi_{65})x_2 - \pi_{35}x_3 = 0 \end{array}$$

and

$$\begin{array}{ll} 1246 & -\pi_{14}x_1 - \pi_{24}x_2 + (\pi_{14} + \pi_{24} + \pi_{64})x_3 = 0 \end{array}$$

are collinear if

$$\begin{vmatrix} 1 & 0 & 0 \\ -\pi_{15} & \pi_{15} + \pi_{35} + \pi_{65} & -\pi_{35} \\ -\pi_{14} & -\pi_{24} & \pi_{14} + \pi_{24} + \pi_{64} \end{vmatrix} = 0$$

By using the relations

$$\sum_i \pi_{i4} = 0, \qquad \sum_i \pi_{i5} = 0 \qquad (i = 1, 2, \ldots, 6)$$

and

$$\pi_{24}\pi_{35} + \pi_{23}\pi_{54} + \pi_{25}\pi_{43} = 0$$

the condition reduces to

$$\pi_{23} + \pi_{43} + \pi_{25} + \pi_{45} = 0$$

The six conditions obtained thus are:

$$\left.\begin{array}{ll} (a) & \pi_{23} + \pi_{43} + \pi_{25} + \pi_{45} = 0 \\ (b) & \pi_{31} + \pi_{41} + \pi_{35} + \pi_{45} = 0 \\ (c) & \pi_{12} + \pi_{42} + \pi_{15} + \pi_{45} = 0 \\ (d) & \pi_{32} + \pi_{42} + \pi_{35} + \pi_{45} = 0 \\ (e) & \pi_{13} + \pi_{43} + \pi_{15} + \pi_{45} = 0 \\ (f) & \pi_{21} + \pi_{41} + \pi_{25} + \pi_{45} = 0 \end{array}\right\} \qquad \text{(I)}$$

As a matter of fact, however, there are only three *independent* conditions upon the π's, the following set of three conditions, for instance, being entirely equivalent* to set (I) above:

$$\left.\begin{array}{ll} (1) & \pi_{41} + \pi_{15} + \pi_{45} = 0 \\ (2) & \pi_{42} + \pi_{25} + \pi_{45} = 0 \\ (3) & \pi_{43} + \pi_{35} + \pi_{45} = 0 \end{array}\right\} \qquad \text{(II)}$$

* Any condition of set (I) may be derived from set (II) (and conversely) by using the relations such as

$$\pi_{12}\pi_{34} + \pi_{13}\pi_{42} + \pi_{14}\pi_{23} = 0$$

and assuming that none of the π's are zero. If one of the π's were zero, we would have a degenerate configuration of no interest.

This set of three conditions is convenient in that they contain only the seven *independent* π's. By adding these three equations, we obtain the useful additional relation

$$(4) \quad \pi_{46} + \pi_{65} - \pi_{45} = 0$$

When conditions (II) are imposed upon the general $\Gamma^4_{6,2}$, the six sets of three lines will be collinear, *and in addition*, the six meeting points $a, b, \ldots, f$ will lie on a conic. Cayley's special $\Gamma^4_{6,2}$ has, then, eleven degrees of freedom instead of the fourteen degrees of freedom of the general $\Gamma^4_{6,2}$.* The general $\Gamma^3_{5,2}$ (Desargues figure) has also just eleven degrees of freedom; and it is interesting to note that any one of the six $\Gamma^3_{5,2}$'s in our Cayley figure may be taken arbitrarily, and that the remainder of the configuration will then be determined. Consider, for instance, the $\Gamma^3_{5,2}$ made up of those elements containing the digit 6. (See Fig. 1.) The two triangles 146, 246, 346 and 156, 256, 356 are perspective, with the point 456 and the line 1236 respectively as center and axis of perspective. The non-corresponding sides of these two triangles meet in six points of a conic, $a, b, \ldots, f$. The sides of the first triangle join the points $a\,e$, $b\,f$ and $c\,d$; those of the second triangle, $a\,f$, $b\,d$ and $c\,e$. If we pair off these letters in the third way cyclic with these two, $a\,d$, $b\,e$ and $c\,f$, and join these pairs of points, we form a third triangle, 145, 245, 345, perspective with the two original triangles from the same center 456. Adding the two new axes of perspective, 1234 and 1235, our Cayley figure is completed.†

Since the π's are co-ordinates in S_4 of the plane of intersection, the three conditions we have found may be interpreted as geometrical restrictions upon the choice of this plane.‡ A linear condition on the π's means that the plane belongs to a linear complex of planes. Our plane must then be one of the ∞^3 planes which are common to three such complexes. In set (I), each of the six conditions indicates that the plane is one of the complex of planes which cut a certain line. Condition (a), for example, indicates that the plane cuts the line common to the three spaces 2345, 1356 and 1246.

---

* Author's paper, *loc. cit.*, Theorem IX.

† Notice that in this Cayley figure there is symmetry with respect to the two triads of letters a, b, c and d, e, f; also with respect to the two triads of digits 1, 2, 3 and 4, 5, 6. In our analytic treatment, the symmetry with respect to 1, 2, 3 is evident, but the symmetry with respect to 4, 5, 6 has been obscured by the special treatment of these digits.

‡ Cayley treated this $\Gamma^4_{6,2}$ as a $C^3_{6,2}$, *i. e.*, as being the projection upon a plane of the figure of six planes in ordinary space. In his first paper, *loc. cit.*, he stated that for this special $C^3_{6,2}$ there were certain conditions upon the six planes; but in the later paper he corrected this statement, and showed that the six planes might be chosen arbitrarily, and that there were three conditions upon the point chosen as center for the projection.

The plane section may of course be obtained by taking first a space section and then a plane section. In this case, the space section may be taken arbitrarily; and then, to obtain the special Cayley figure, the cutting plane is determined. In the intermediate space configuration $\Gamma^4_{6,3}$ (the figure of two perspective tetrahedra), the combinations 2345, 1356, and 1246 represent planes. Let the point determined by these three planes be called a. Five other points, $b, c, \ldots, f$, may be similarly determined in accordance with the table in paragraph 1. These six points lie in a plane, which we use as the cutting plane. Ten such planes are determined by any $\Gamma^4_{6,3}$ configuration, corresponding to the ten ways of separating the six digits into two triads.

3. *The Conics ϕ of the Cayley Figure.* We naturally expect to find some connection between the conic C and the pencil of conics ϕ. Using the relations of set (II), the equations of the conics ϕ_4 and ϕ_5 reduce to

$$\phi_4 \equiv \sum_i \pi_{i4}\,\pi_{i5}\,x_i^2 + 2\sum_{ij} \pi_{i4}\,\pi_{j4}\,x_i\,x_j = 0$$
$$\phi_5 \equiv \sum_i \pi_{i5}\,\pi_{i4}\,x_i^2 + 2\sum_{ij} \pi_{i5}\,\pi_{j5}\,x_i\,x_j = 0$$

The co-ordinates of the six points $a, b, \ldots, f$ are found to be

$$\begin{array}{llll} a: & (0,\ \pi_{53},\ \pi_{24}) & d: & (0,\ \pi_{43},\ \pi_{25}) \\ b: & (\pi_{34},\ 0,\ \pi_{51}) & e: & (\pi_{35},\ 0,\ \pi_{41}) \\ c: & (\pi_{52},\ \pi_{14},\ 0) & f: & (\pi_{42},\ \pi_{15},\ 0) \end{array}$$

and the conic C, through these points, is simply

$$C \equiv 2\sum_i \pi_{i4}\,\pi_{i5}\,x_i^2 + 2\sum_{ij} (\pi_{i4}\,\pi_{j4} + \pi_{i5}\,\pi_{j5})\,x_i\,x_j \equiv \phi_4 + \phi_5 = 0$$

which shows that the conic C lies in the pencil with the conics ϕ. Any conic of this pencil may be expressed in the form

$$\phi_4 + \lambda\,\phi_5 = 0$$

the three conics ϕ_4, ϕ_5 and C becoming thus the *base* conics of the pencil, *i. e.*, the conics having the parameters 0, ∞ and 1 respectively.

If λ_i denote the parameter of the conic ϕ_i, we have

$$\lambda_4 = 0, \quad \lambda_5 = \infty, \quad \text{and } \lambda_i = \frac{\pi_{i4}}{\pi_{i5}} \text{ (when } i = 1, 2, 3 \text{ or } 6)$$

and hence we have the relation

$$\frac{1}{\lambda_1 - 1} + \frac{1}{\lambda_2 - 1} + \frac{1}{\lambda_3 - 1} = \frac{1}{\lambda_4 - 1} + \frac{1}{\lambda_5 - 1} + \frac{1}{\lambda_6 - 1}$$

If the λ's were parameters of points on a line, this equation would indicate that the point whose parameter was 1 had the same second polar with respect to

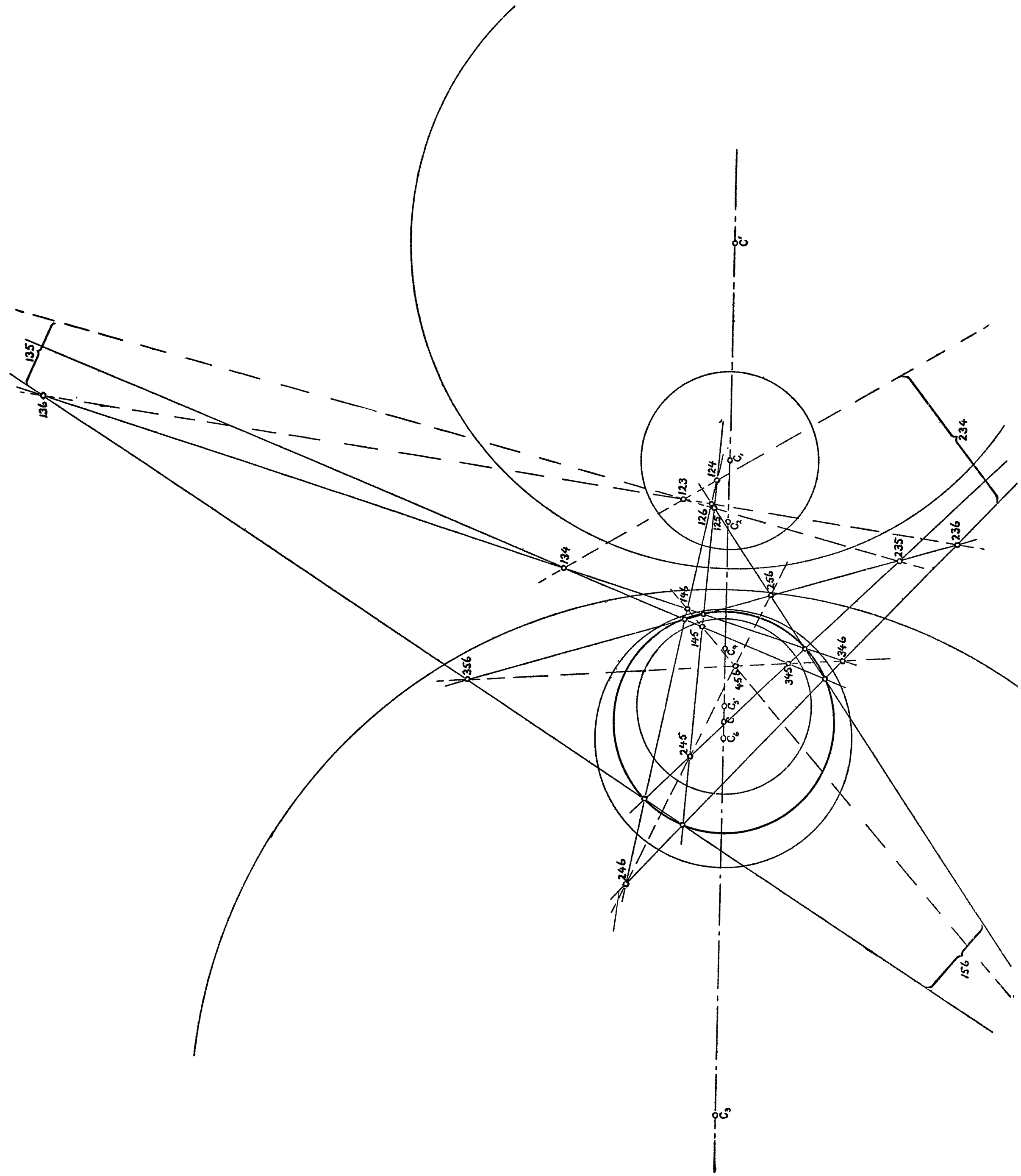

Fig. 2.

The conics C_2 and C_4 are imaginary, but their centers are shown in the figure.

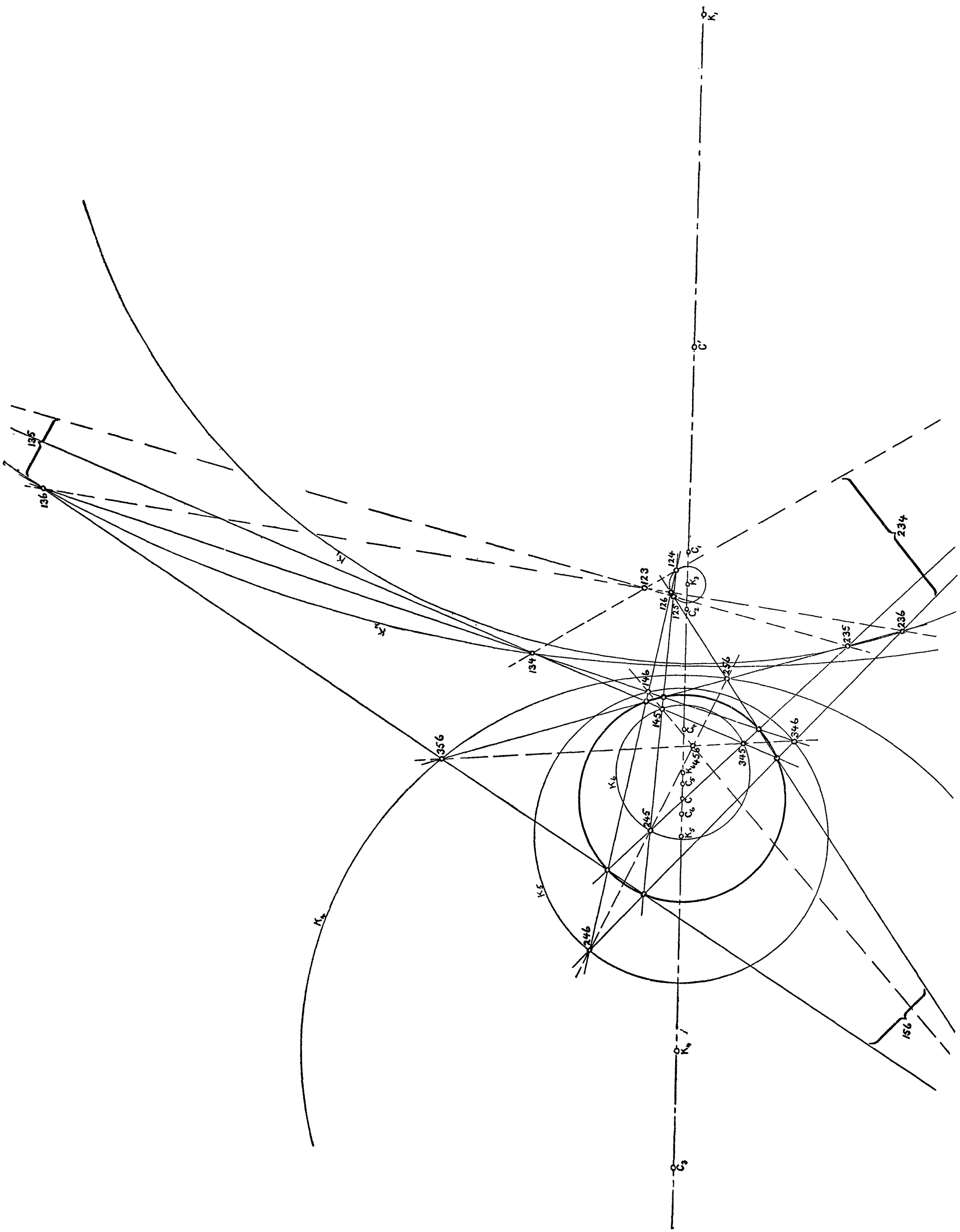
136
135
123
124
126
134
356
146
145
256
346
345
245
246
234
235
236
156

FIG. 3.

the three points λ_1, λ_2, λ_3 that it had with respect to the points λ_4, λ_5, λ_6.* Considering the conics of our pencil, then, as elements, we may say that C has the same second polar with respect to the two triads of conics ϕ_1, ϕ_2, ϕ_3 and ϕ_4, ϕ_5, ϕ_6. The conic C' which is the common second polar of C with respect to the two triads has the parameter (see Fig. 2)

$$\lambda = \frac{\pi_{65} - 3\,\pi_{45}}{\pi_{65} + \pi_{45}}$$

4. *The Conics k.* We next notice that the conic

$$\phi_4 - \phi_5 = 0$$

has no square terms, and is therefore circumscribed to the triangle 145, 245, 345. This is one of the three triangles which are in perspective from the point 456. Because of the peculiar symmetry of our configuration, it follows that there is a conic of the pencil which is circumscribed to each of the six triangles

(1)	234, 235, 236	(4)	156, 256, 356
(2)	134, 135, 136	(5)	146, 246, 346
(3)	124, 125, 126	(6)	145, 245, 345

and these six conics may be designated respectively (see Fig. 3) as

$$k_1,\ k_2,\ k_3,\ k_4,\ k_5 \text{ and } k_6$$

We have seen that the parameter of k_6 is -1. Hence we may say that k_6 is the polar of C with respect to ϕ_4 and ϕ_5. Similarly k_5 must be the polar of C with respect to ϕ_4, ϕ_6; k_1 the polar of C with respect to ϕ_2, ϕ_3; etc. The parameter λ for each of the conics in the pencil is shown in the following table:

Conic	Parameter	Conic	Parameter
C	1	C'	$\dfrac{\pi_{65} - 3\,\pi_{45}}{\pi_{65} + \pi_{45}}$
ϕ_1	$\dfrac{\pi_{14}}{\pi_{15}}$	k_1	$\dfrac{\pi_{24} + \pi_{34}}{\pi_{25} + \pi_{35}}$
ϕ_2	$\dfrac{\pi_{24}}{\pi_{25}}$	k_2	$\dfrac{\pi_{34} + \pi_{14}}{\pi_{35} + \pi_{15}}$
ϕ_3	$\dfrac{\pi_{34}}{\pi_{35}}$	k_3	$\dfrac{\pi_{14} + \pi_{24}}{\pi_{15} + \pi_{25}}$
ϕ_4	0	k_4	$\dfrac{2\,\pi_{64} - \pi_{65}}{\pi_{65}}$
ϕ_5	∞	k_5	$\dfrac{\pi_{64}}{2\,\pi_{65} - \pi_{64}}$
ϕ_6	$\dfrac{\pi_{64}}{\pi_{65}}$	k_6	-1

* These λ's may, for instance, be considered as the parameters of the centers of the conics. These centers have the polar relations as stated (see Fig. 2).

5. *The Common Tangents to Pairs of the Conics k.* It will be seen that eighteen of the twenty points of the configuration — all except the points 123 and 456 — lie on the conics k, three points on each of the six conics. Opposite points of the configuration, such as 124 and 356, are conjugate with respect to each conic of the pencil.* Hence the line joining 124 and 356 is cut by these conics in a quadratic involution of which 124 and 356 are the double points. But since these points lie respectively on k_3 and k_4, they must be the points of contact of a common tangent to these two conics. Similarly, all the points of the configuration except 123 and 456 are points of contact of common tangents to two conics, one from each of the two sets k_1, k_2, k_3 and k_4, k_5, k_6.

If we draw tangents to the conics k_4, k_5 and k_6 respectively at the points 156, 146 and 145, these tangents all touch the conic k_1. The points 156, 146 and 145 lie on the straight line 1456. There must, of course, be some relation between four conics of a pencil in order that such an arrangement be possible. To discover this relation, we may consider the metrically special case of a pencil of circles. Using rectangular Cartesian co-ordinates, and taking the pencil

$$(x^2 + y^2 + 2ax + a^2) + \lambda(x^2 + y^2 - 2ax + a^2) = 0$$

the base circles of the pencil will be the three degenerate circles. By taking the pencil in this way, any *projective* relation between circles of the pencil will be given by a condition on the λ's independent of the constant a, and the result will be applicable to any pencil of conics. The common tangent to the two circles k_1 and k_4 touches k_4 at the point

$$\left(k\,\frac{\sqrt{\lambda_4} + \sqrt{\lambda_1}}{\sqrt{\lambda_4} - \sqrt{\lambda_1}} \qquad 2k\,\frac{\sqrt{-\lambda_4(1-\lambda_1)}}{(\sqrt{\lambda_4} - \sqrt{\lambda_1})\sqrt{1-\lambda_4}} \right)$$

and the condition that the three such points for k_4, k_5 and k_6 lie on a straight line is

$$\begin{vmatrix} 1 & \dfrac{1}{\sqrt{\lambda_4}} & \dfrac{1}{\sqrt{1-\lambda_4}} \\ 1 & \dfrac{1}{\sqrt{\lambda_5}} & \dfrac{1}{\sqrt{1-\lambda_5}} \\ 1 & \dfrac{1}{\sqrt{\lambda_6}} & \dfrac{1}{\sqrt{1-\lambda_6}} \end{vmatrix} = 0$$

* Author's paper, *loc. cit.*, p. 544.

When we remove the factors* which simply make $\lambda_4 = \lambda_5$, etc., we have left the condition

$$(\sqrt{\lambda_5} + \sqrt{\lambda_6})\sqrt{1-\lambda_4} + (\sqrt{\lambda_6} + \sqrt{\lambda_4})\sqrt{1-\lambda_5} + (\sqrt{\lambda_4} + \sqrt{\lambda_5})\sqrt{1-\lambda_6} = 0 \qquad \text{(III)}$$

It is noticeable that this condition is entirely free from λ_1. A similar relation must exist, of course, between λ_1, λ_2, and λ_3.

It is interesting to build up the configuration from these conics k. The three degenerate conics of any pencil are chosen; and then the conics k_4, k_5, and k_6 of the pencil are taken with their parameters (referred to the degenerate conics) satisfying condition (III). k_1 may then be taken as any other conic of the pencil. *Any one* of the four common tangents to k_1 and k_4 may be drawn, the points of contact being the points 234 and 156. Then we can select *one* of the common tangents to k_1 and k_5, and *one* of the common tangents to k_1 and k_6, so that the points of contact 146 and 145 on k_5 and k_6 respectively will lie on a line with 156. The sixteen possible selections after the first common tangent is drawn correspond to the sixteen possible choices of sign in condition (III). The remaining points of the configuration will now be linearly determined, together with all the other conics. Although there are thus four different configurations determined when the conics k_1, k_4, k_5 and k_6 are fixed, the conics k_2 and k_3 are the same for each of the four cases, and hence are *uniquely* determined by k_4, k_5, k_6 and k_1. It has been shown (Part I, Section 3) that when the six conics ϕ are given, four different $\Gamma_{6,2}^4$'s are determined by them. If, in this special $\Gamma_{6,2}^4$, the k's are given instead of the ϕ's, the *same* four $\Gamma_{6,2}^4$'s are determined.

If we consider the metrically special case in which the ϕ's and k's are circles, it is evident that when one of the four configurations is drawn, another of the four may be obtained by reflecting the whole figure in the line of centers of the circles. This fact may be readily translated into projective language. Let the conics ϕ and k be given, and let one of the four configurations which they determine be drawn. If a point of the configuration lying upon k_i be projected upon k_i from one of the vertices of the common self-polar triangle of the conics, the point obtained will be the corresponding point of one of the other

* If we make the substitution $\lambda_i = \sin^2 \alpha_i$, we can readily remove the factors $\sin\frac{\alpha_5 - \alpha_6}{2}$, $\sin\frac{\alpha_6 - \alpha_4}{2}$ and $\sin\frac{\alpha_4 - \alpha_5}{2}$ from the determinant.

configurations. Each vertex of the self-polar triangle may thus be used to obtain the points of one of the configurations. The points a, b, c, d, e, and f on the conic C will be similarly projected into three other sets—a set corresponding to each configuration.

The Pascal hexagram contains ten of these Cayley configurations, each with its pencil of conics. The conic C lies in all of the ten pencils. The conics of any one of these ten pencils determine three new Cayley configurations and three new sets of six points, $a, b, \ldots, f$, on C. The three new sets determined by the conics of one pencil are *not the same* as the three sets determined by the conics of another. It would be interesting to discover just how this extensive system closes in.

CORNELL UNIVERSITY, *June*, 1907.